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UNIQUENESS OF THE EXTERNAL PROBLEMS SOLUTION IN THE THEORY OF ELASTIC VIBRATIONS OF ANISOTROPIC MEDIA

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There is considered the question regarding the extraction of a unique solution, defined in all space, of the inhomogeneous problem for the system of equations for steady vibrations of elastic media in plane strain on the basis of the radiation principle. There are presented conditions at infinity, which go over directly into the Sommerfeld condition upon making the transition to isotropic media.

1. Let us consider the system of equations

$$c_1 \frac{\partial^2 u}{\partial x_1^2} + c_2 \frac{\partial^2 w}{\partial x_1 \partial x_2} + c_3 \frac{\partial^2 u}{\partial x_2^2} - \rho \frac{\partial^2 u}{\partial t^2} = 0 \tag{1.1}$$

$$c_3 \frac{\partial^2 w}{\partial x_1^2} + c_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} + c_4 \frac{\partial^2 w}{\partial x_2^2} - \rho \frac{\partial^2 w}{\partial t^2} = 0$$

describing the vibrations of cubic, hexagonal, certain tetragonal, rhombic, and rhombohedric crystals as well as orthotropic and transversely isotropic media in plane strain. Here u, w are components of the displacements along the axes x and x_2, the constants c_i are expressed in terms of the elastic constants of the medium [1], t is the time, and ρ is the density.

The system (1.1) is strictly hyperbolic under the conditions

$$-2\sqrt{\alpha\beta} < \gamma < 1 + \alpha\beta \tag{1.2}$$

$$(\alpha = c_3 / c_1,\ \beta = c_3 / c_4,\ \gamma = 1 + \alpha\beta - c_2^2 / c_1 c_4)$$

The positive-definiteness conditions for the elastic energy have a different form depending on the specific form of the elastic symmetry and relate both the elastic constants in c_i (four), and those not in c_i in all cases, except the case of cubic crystals [1].

The characteristic polynomial of the system (1.1) is written in the form

$$\Delta = (\rho - c_3\mu^2 - c_1\theta^2)(\rho - c_4\mu^2 - c_3\theta^2) - c_2^2\theta^2\mu^2 \tag{1.3}$$

and is a fourth-order polynomial in θ and μ.

The real zeroes of the characteristic polynomial (1.3) form two closed (internal and external) non-intersecting curves on the plane θ, μ which are symmetric with respect to the coordinate axes. In the particular case $\alpha = \beta$, the curves of the real zeroes of the characteristic polynomial are divided into four kinds [1] depending on the value of γ

$$1)\ \gamma_* < \gamma < 1 + \alpha^2, \quad 2)\ 2\alpha < \gamma < \gamma_* \tag{1.4}$$

$$3)\ \alpha(\alpha + 1) < \gamma < 2\alpha, \quad 4)\ -2\alpha < \gamma < \alpha(\alpha + 1)$$

where only $\alpha < 1$ are admissible [1]. Here

$$2\gamma_* = -3(1-\alpha)^2 + (1+\alpha)\sqrt{9\alpha^2 - 14\alpha + 9}$$

All four configurations are presented in Fig. 1. Isotropic media correspond to the value $\gamma = 2\alpha$ when the curves of the real zeroes are concentric circles.

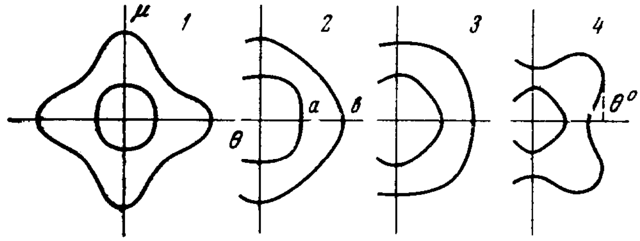

Fig. 1

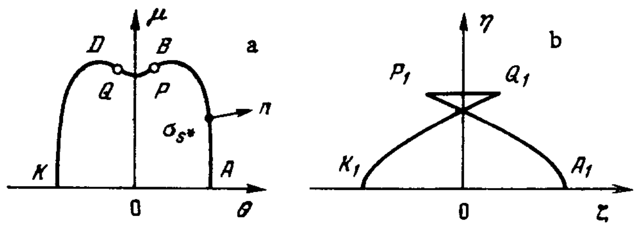

Fig. 2

The characteristic equation $\Delta(\theta, \mu) = 0$ has four roots. In the case $\alpha = \beta$ we can write

$$\mu_{1,3} = \pm (N_1 - \sqrt{N_1^2 - N_2})^{1/2}, \quad \mu_{2,4} = \pm (N_1 + \sqrt{N_1^2 - N_2})^{1/2} \qquad (1.5)$$
$$2N_1 = (1+\alpha)\, b^2 - \theta^2(\gamma / \alpha), \quad b^2 = \rho c_3{}^{-1}$$
$$N_2 = (a^2 - \theta^2)(b^2 - \theta^2), \quad a^2 = \rho c_1{}^{-1}$$

The functions $\mu_n(\theta)$, examined at an arbitrary point of the real axis θ, can also take on imaginary value (in the first two of the four cases in (1.4)) in addition to the real values (points on the curves of real zeroes), or pure imaginary and complex conjugate values (the last two cases in (1.4)).

For a single-valued determination of the four-valued function $\mu(\theta)$, which is a solution of the characteristic equation, it is necessary to construct a four - sheeted Riemann surface above the plane of the complex variable θ. We construct the Riemann surface by starting from the following considerations. Any line $\theta = \text{const}$ in the interval $|\theta| < a$ (as well as in the interval $b < |\theta| < \theta^\circ$ in the fourth case of (1.4)) intersects the curves of real zeroes at four real points. At two points the external normal to the curves has a positive projection on the μ -axis, and a negative projection on the other two. The locus of points of the first kind will be denoted by μ_1 and μ_2. Two values of the function $\mu(\theta)$, denoted by $\mu_1(\theta)$ and $\mu_2(\theta)$ (the first and second sheets of the Riemann surface) will correspond to them. We denote the locus of points of the second kind by μ_3 and μ_4. We denote the two values of the function $\mu(\theta)$ corresponding to them by $\mu_3(\theta) = -\mu_1(\theta)$ and $\mu_4(\theta) = -\mu_2(\theta)$(the third and fourth sheets). The separate sheets are interconnected by means of slits drawn through branch points of the external and internal radicals in (1.5).

2. At time $t = 0$, let a point-pulsed source of perturbations be activated at the origin $x_1 = x_2 = 0$ in an infinite elastic medium. Assuming that the medium is at rest prior to the time $t = 0$, we obtain that quasi-longitudinal and quasi-transverse perturbation waves will be propagated in all directions from the point $x_1 = x_2 = 0$. The wave fronts are envelopes of families of lines $t - \theta x_1 - \mu_l(\theta) x_2 = 0$, $l = p, s$, and are determined from the solutions of the system of equations which we write in the form

$$t = \mu_l(\theta) r \sin\varphi + \theta r \cos\varphi, \quad \mu_l'(\theta)\sin\varphi + \cos\varphi = 0 \qquad (2.1)$$
$$(x_1 = r\cos\varphi, \ x_2 = r\sin\varphi)$$

The values of $\mu_l(\theta)$ here refer to points on the curves of real zeroes.

The second equation in (2.1) determines a point on the curve of real zeroes S_l at which

$$\mu_l'(\theta) = -\operatorname{ctg}\varphi, \quad \theta = \theta_{l_*} \qquad (2.2)$$

Substituting $\theta = \theta_{l_*}$ in the first of equations (2.1), we find t_l corresponding to the residence time of the perturbation at the point (x_1, x_2).

We refer the value $l = p$ to points on the inner curve and $l = s$ to points on the outer curve of real zeroes.

It follows from (2.2) that if the curves of real zeroes are convex, then the wave fronts are convex closed curves without angular points. If the outer curve of real zeroes has an inflection point, then the wave front will have angular points (cusps of the first kind). For the configuration presented in Fig. 2a, the wave fronts will have the form shown in Fig. 2b (this configuration is possible in the case $\alpha \neq \beta$) where we use the notation $\xi = x_1 / t$ and $\eta = x_2 / t$, $\eta > 0$.

Substituting the value θ_{l_*} in the first equation of (2.1), we find

$$v_l^{-1}(\varphi) \equiv t_l / r = \mu_{l_*}\sin\varphi + \theta_{l_*}\cos\varphi, \quad \mu_{l_*} = \mu_l(\theta_{l_*})$$

It hence follows that the quantity reciprocal to the velocity $v_l(\varphi)$ with which the perturbation occurring at the origin at the time $t = 0$ arrives at the point of observation (x_1, x_2), equals the projection of the vector drawn from the origin to the point σ_{l_*} on the curve of real zeroes S_l $(\sigma_{l_*} = (\theta_{l_*}, \mu_{l_*}))$ in the $\theta\mu$ plane, at which the vector of the normal coincides, in direction, with the vector $x = (x_1, x_2)$ in the direction of this normal. The velocity v_l is called the ray velocity.

There is always just one point σ_{p_*} on the inner curve of real zeroes S_p since the inner curve is always convex. Denoting quantities reciprocal to the ray velocities by $c_l(\varphi)$, $l = p, s$, we have

$$c_p(\varphi) = \mu_{p_*}\sin\varphi + \theta_{p_*}\cos\varphi \qquad (2.3)$$

The outer curve S_s can be convex; then there is just one point σ_{s_*} and one value, respectively, $c_s(\varphi)$ for a given φ

$$c_s(\varphi) = \mu_{s_*} \sin \varphi + \theta_{s_*} \cos \varphi \tag{2.4}$$

or there are concavity sections (inflection point). In this latter case, there is either just one point σ_{s_*} depending on the value of φ (for values of φ corresponding to rays not passing through the lacunas) and one value $c_s(\varphi)$ defined by (2.4), or three points $\sigma_{s_*}^{j}$ $(j = 1, 2, 3)$ at which the directions of the normals coincide with the direction of the vector x; we have

$$c_s^{j}(\varphi) = \mu_{s_*}^{j} \sin \varphi + \theta_{s_*}^{j} \cos \varphi, \quad j = 1, 2, 3$$

3. Let us be interested in solutions of the form

$$u = u_1 \exp(igt), \quad w = u_2 \exp(igt)$$

We obtain the system of equations

$$L_{it}(u_i) = 0, \quad i, t = 1, 2 \tag{3.1}$$

$$L_{11} = c_1 \frac{\partial^2}{\partial x_1^2} + c_3 \frac{\partial^2}{\partial x_2^2} + p^2, \quad L_{21} = c_2 \frac{\partial^2}{\partial x_1 \partial x_2}$$

$$L_{22} = c_3 \frac{\partial^2}{\partial x_1^2} + c_4 \frac{\partial^2}{\partial x_2^2} + p^2, \quad L_{12} = L_{21}, \quad p^2 = \rho g^2$$

for the stationary part of the displacements u_1 and u_2 which will be of elliptic type for the conditions (1.2).

Let us examine the domain G of the space $x_1 x_2$ with boundary B. Let the functions u_1, u_2 in the domain G be continuously differentiable twice and satisfy the inhomogeneous system of equations

$$L_{it}(u_i) = F_t \tag{3.2}$$

and the functions v_1^{j} and v_2^{j} $(j = 1, 2)$ twice continuously differentiable everywhere except at the points $(x_1^{\circ}, x_2^{\circ})$ and satisfy the system of equations

$$L_{it}[v_i^{j}(x, x^{\circ})] = \delta_i^{j}\, \delta(x - x^{\circ})$$

$$(x = (x_1, x_2), \; x^{\circ} = (x_1^{\circ}, x_2^{\circ}))$$

where δ_i^{j} is the Kronecker symbol. Then from the Green 's formula for the self-adjoint operator L_{it} [2]

$$\int_G \{u_i L_{it}(v_t) - v_t L_{it}(u_i)\}\, dx = \int_B M_{it}(u_i, v_t)\, ds, \quad dx = dx_1\, dx_2$$

we obtain

$$u_j(x^{\circ}) = \int_G v_t^{j}(x) F_t(x)\, dx + \int_B M_{it}[u_i(x), v_t^{j}(x, x^{\circ})]\, ds \tag{3.3}$$

It is assumed that F_t are sufficiently smooth functions, equal to zero outside a certain domain lying entirely within the domain G. The expression for M_{it} is written in the form

$$M_{it}(u_i, v_t) = U\,T(V) - VT(U)$$
$$UT(V) = M_1(u_i, v_i)\cos\psi + M_2(u_i, v_i)\sin\psi$$
$$VT(U) = M_1(v_i, u_i)\cos\psi + M_2(v_i, u_i)\sin\psi$$
$$M_1(u_i, v_i) = u_1\sigma_{11}(v_1, v_2) + u_2\tau_{12}(v_1, v_2)$$
$$M_2(u_i, v_i) = u_1\tau_{21}(v_1, v_2) + u_2\sigma_{22}(v_1, v_2)$$

where ψ is an angle which the external normal to the contour makes with the x_1 axis, $\sigma_{11} = \sigma_{x_1}$, $\tau_{12} = \tau_{x_1x_2}$, $\tau_{21} = \tau_{x_2x_1}$, $\sigma_{22} = \sigma_{x_2}$ are stress operators acting on the space of vector-functions indicated in parentheses. The functions v_t^j form a fundamental matrix of the operator L_{it}.

Let B be a circle of radius R with center at the origin. The sources defined by the functions F_t are in the finite part of the plane lying entirely within the domain G. Let the domain G increase without limit, by letting R tend to infinity. Then, for the sources given in the finite part of the plane to define uniquely a solution of the problem in the case of an infinite medium, the following additional condition should be satisfied.

$$\lim_{R\to\infty}\int_B M_{it}[u_i, v_t^j]\,ds = 0 \tag{3.4}$$

The first integral in (3.3) is a particular solution u_j^1 of the inhomogeneous problem since it is the convolution of the right side and the fundamental solution. We obtain that as $R \to \infty$

$$u_j' \equiv \int_B M_{it}[u_i, v_t^j]\,ds = u_j - u_j^1$$

as the difference between two solutions of the inhomogeneous problem, is a solution of the homogeneous problem, i.e., satisfies the system $L_{jt}(u'_j) = 0$ in the whole plane.

To assure the uniqueness of the solution of the inhomogeneous problem in the whole plane, it is sufficient to impose such conditions on the function u_i at infinity that the solutions of the homogeneous problem satisfying them could only occur by identical means.

4. Uniqueness theorem. Let the functions u_i be 1) twice continuously differentiable and satisfy the homogeneous system of equations $L_{it}(u_i) = 0$ in the whole plane $x = (x_1, x_2)$, and 2) representable in the form $u_i = u_i^p + u_i^s$, where

a) The functions u_i^p satisfy the following conditions in the neighborhood of infinity

$$u_i^p = O(r^{-1/2}), \quad \frac{\partial u_i^p}{\partial r} - ik_p(\varphi)\,u_i^p = o(r^{-1/2}) \tag{4.1}$$

b) In the case of strictly convex curves of real zeroes for any φ and in the case when there are concavity sections on the curves of real zeroes, the functions u_i^s will satisfy conditions of the form

$$u_i^s = O(r^{-1/2}), \quad \frac{\partial u_i^s}{\partial r} - ik_s(\varphi)\,u_i^s = o(r^{-1/2}) \tag{4.2}$$

at infinity in angular domains of the first kind;

c) The functions u_i^s will satisfy the conditions

$$u_i^s = \sum_{j=1}^{3} u_{ij}^s, \quad u_{ij}^s = O(r^{-1/2}), \quad \frac{\partial u_{ij}^s}{\partial r} - ik_s^j(\varphi)\, u_{ij}^s = o(r^{-1/2}) \tag{4.3}$$

at infinity in angular domains of the second kind.

Then the functions u_i equal zero identically in the whole x_1x_2 plane.

Here

$$k_p(\varphi) = gc_p(\varphi), \quad k_s(\varphi) = gc_s(\varphi), \quad k_s^j(\varphi) = gc_s^j(\varphi)$$

The x_1x_2 plane is partitioned as follows when there are concavity sections in the angular domains of the first and second kind on the outer curve of normals. We draw the rays $\varphi_\nu(\nu = 1,2,\ldots, m$ (here m equals the number of points on the curve of real zeroes where the curvature is zero) from the origin at angles equal to the slopes of the normals to the curves of real zeroes at points where the curvature equals zero (inflection points). These rays will pass through angular points on the fronts. We refer angular domains without lacunae to the angular domains of the first kind. The angular domains in which lacunae exist are referred to the domains of the second kind. The wave pattern with the mentioned partition of the x_1x_2 plane into angular domains is presented in Fig. 3 for media with $\alpha = \beta$, $-2\alpha < \gamma < \alpha(\alpha + 1)$.

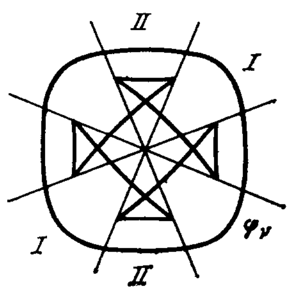

Fig. 3

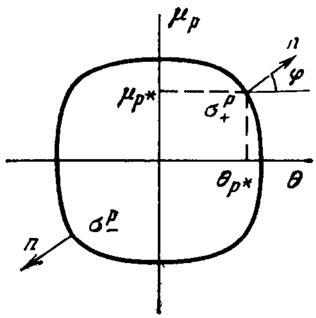

Fig. 4

Let us first examine the case when the curves of real zeroes are strictly convex. If $\alpha = \beta$ then this is a medium with $2\alpha < \gamma < \gamma_*$, $\alpha(\alpha + 1) < \gamma < 2\alpha$. The boundary value $\gamma = 2\alpha$ refers to isotropic media. In the general case $\alpha \neq \beta$, and $0 < \alpha < 1$, $0 < \beta < 1$ the conditions

$$\alpha(\beta + 1) < \gamma < \gamma_*^\circ \quad \text{for} \quad \alpha > \beta$$
$$\beta(\alpha + 1) < \gamma < \gamma_*^\circ \quad \text{for} \quad \beta > \alpha$$

refer to these media.

The quantity $\gamma_*^\circ = \gamma_*(\alpha, \beta)$ is presented in [3].

Let us consider a differential equation of the form

$$P(D)\,u(x) = f(x) \qquad (4.\;)$$

Here $P(\sigma_1, \ldots, \sigma_n)$ is some polynomial, $x = (x_1, \ldots, x_n)$ is a point in n-dimensional space, D denotes the formal vector $i^{-1}(d/dx_1, \ldots, d/dx_n)$, and σ denotes the point $(\sigma_1, \ldots, \sigma_n)$.

Let us assume the following conditions to be satisfied:

a) $P(\sigma)$ is a hypoelliptic polynomial;

b) All coefficients of the polynomial $P(\sigma)$ are real numbers;

c) For all real σ for which $P(\sigma) = 0$, the condition $\operatorname{grad} P(\sigma) \neq 0$ is satisfied;

d) The curvature is different from zero at each point of the surface of real zeroes of the polynomial.

There results from conditions b) and c) that all the real zeroes of the polynomial $P(\sigma)$ are located on one or more smooth closed surfaces of the dimension $n - 1$. Let them be denoted by $S_1, \ldots, S_k$.

Let us consider one of the surfaces S_l. Let $\sigma_+^l(\omega)$ denote a point on the surface S_l at which the normal to S_l directed outward from S_l parallel to ω (ω is a unit vector, $x = \omega r$), and $\sigma_-^l(\omega)$ a point at which the normal has the opposite direction. These points are uniquely defined since each of the surfaces S_l is strictly convex (Fig.4).

Let $Q(\omega, D)$ be a certain fixed differential operator whose coefficients depend only on ω such that for any ω and $1 \leqslant l \leqslant k$ the polynomial $Q(\omega, \sigma)$ is zero at one of the points $\sigma_+^l(\omega)$ and $\sigma_-^l(\omega)$ and different from zero at the other. For $x \neq 0$ the coefficients of the operator $Q(\omega, D)$ are assumed sufficiently smooth.

The following uniqueness theorem is proved in [4]: If the polynomial $P(\sigma)$ satisfies conditions a) — d), then the solution of the homogeneous equation

$$P(D)\,u(x) = 0 \qquad (4.5)$$

satisfying the conditions

$$u(x) = o\,(r^{(3-n)/2}), \quad Q(\omega, D)\,u(x) = o\,(r^{(1-n)/2}) \qquad (4.6)$$

$$Q[\omega, \sigma_\pm^l(\omega)] = 0, \quad Q[\omega, \sigma_\mp^l(\omega)] \neq 0, \quad 1 \leqslant l \leqslant k \qquad (4.7)$$

must be identically zero. Here the upper and lower signs are taken in the subscripts depending on the selection of the surface S_l.

Any two polynomials $Q_1(\omega, \sigma)$ and $Q_2(\omega, \sigma)$, which vanish simultaneously at one of the points $\sigma_+^l(\omega)$ or $\sigma_-^l(\omega)$ and not at the other, extract the same solution of (4.5) by using conditions (4.6) and (4.7).

According to [5], every solution of a homogeneous system of linear equations with constant coefficients is also a solution of some scalar equation. In the case under consideration the equation is written in the form

$$P(D)u = 0, \quad P(D) = L_{11}L_{22} - L_{12}L_{21} \qquad (4.8)$$

where $P(D)$ is a fourth order differential operator. To prove the theorem, it is sufficient to show that conditions (4.1) and (4.2) are of the same nature as conditions (4.6) and (4.7).

Rewriting the expressions in the right in (4.1) and (4.2) in the form

$$\left[\frac{\partial}{\partial r} - ik_l(\varphi)\right] u_i^l = o(r^{-1/2}), \quad l = p, s \tag{4.9}$$

and going over from differentiation with respect to the radius to differentiation with respect to x_1 and x_2, we obtain

$$Q_l(\omega, D) \equiv \frac{\partial}{\partial r} - ik_l(\varphi) = \frac{\partial}{\partial x_1}\cos\varphi + \frac{\partial}{\partial x_2}\sin\varphi - ik_l(\varphi)$$

Moreover, replacing the formal vector $i^{-1}(\partial / \partial x_1, \partial / \partial x_2)$ by the vector $\sigma = (\sigma_1, \sigma_2)$, $\sigma_1 = g\theta$, $\sigma_2 = g\mu$, we will have

$$Q_l(\omega, \sigma) = ig(\mu \sin\varphi + \theta\cos\varphi) - ik_l(\varphi), \quad l = p,s$$

The polynomial

$$Q_1(\omega, \sigma) = Q_p(\omega, \sigma)\, Q_s(\omega, \sigma)$$

can be taken as $Q(\omega, \sigma)$.

It is seen that $Q_1(\omega, \sigma_+^l) = 0$ while $Q_1(\omega, \sigma_-^l) \neq 0$ for $l = p, s$ if each of the polynomials $Q_l(\omega, \sigma)$ vanishes at the appropriate point σ_+^l and does not at σ_-^l.

We find

$$Q_l(\omega, \sigma_+^l) = 0, \quad Q_l(\omega, \sigma_-^l) = -i2k_l(\varphi)$$

at the points $\sigma_+^l(\omega), \sigma_-^l(\omega)$ (Fig. 4), whose coordinates equal $(\theta_{l*}, \mu_l(\theta_{l*}))$ and $(-\theta_{l*}, -\mu_l(\theta_{l*}))$ respectively.

Here $l = p$ refers to the inner and $l = s$ to the outer curve of real zeroes. Therefore, conditions (4.6) and (4.7) are satisfied, and since the polynomial $P(\sigma)$, defined by (4.8), satisfies all the hypotheses a) — d), the uniqueness theorem for the system (3.2) results from the theorem for (4.5), where $P(\sigma)$ is defined by (4.8).

Taking account of the asymptotic form of the solutions of the inhomogeneous problem as $r \to \infty$ [6] in the case of convex curves of real zeroes, conditions (4.1) and (4.2) can be written in the form

$$u_i^p = O(r^{-1/2}), \quad \frac{\partial u_i^p}{\partial r} - ik_p(\varphi)\, u_i^p = O(r^{-3/2})$$

$$u_i^s = O(r^{-1/2}), \quad \frac{\partial u_i^s}{\partial r} - ik_s(\varphi)\, u_i^s = O(r^{-3/2})$$

When the outer curve of normals has an inflection point, the wave fronts have acute-angled edges (the points P_1 and Q_1 in Fig. 2), and the directions of the radii of the vectors passing through the angular points (P_1 and Q_1) are singular. In the neighborhood of these directions, the asymptotic of the solutions of the homogeneous problem will not be uniform in φ as $r \to \infty$ and the remainder term of the asymptotic expansions does not equal $O(r^{-3/2})$ as in the case of the ordinary direction (or for any φ in the case of convex curves of real zeroes). Using the asymptotic expansions for the neighborhood of the singular directions presented in [6], the proof of the theorem for the case of curves of real zeroes having inflection points can be carried out not by following [4] but by direct estimation of the integral (3.4) by the method elucidated in [7].

From the viewpoint of a physical description of the process of steady propagation of vibrations in an anisotropic elastic medium, conditions (4.1) – (4.3) indicate that the solution of every problem about steady vibrations of an infinite medium (independently of the specific form and location of the sources given in the finite part of space) will degenerate at infinity into traveling diverging waves, where the equal-phase curves agree with the fronts of the waves being propagated from a concentrated pulse source. The distinction from isotropic media is that the ray velocity in the anisotropic medium depends on the direction and the equal-phase curves will not be circles. In going over to $\alpha = \beta$ and $\gamma = 2\alpha$ (isotropic medium), we have

$$c_p(\varphi) = a, \quad c_s(\varphi) = b, \quad k_p(\varphi) = k, \quad k_s(\varphi) = l$$

$$k = ga, \quad l = gb, \quad a = \sqrt{\rho / c_1}, \quad b = \sqrt{\rho / c_3}$$

and conditions (4.1) and (4.2) go over into the known Sommerfeld condition for the Helmholtz equations. At infinity the equal-phase curves will be concentric circles.

Each of the conditions (4.1) and (4.2) discerns a definite class of uniqueness of solutions of the system (3.2). Hence, the inhomogeneous system of equations (3.2), for a given right side, allows one solution from the class W^p formed by the functions u_i^p, and one solution from the class W^s, formed by the functions u_i^s. This means that two kinds of waves, quasi-longitudinal and quasi-transverse, can be propagated in a medium.

In the case under consideration of the whole space (the plane x_1x_2), each of the kinds of waves can be propagated separately and independently of the other. Hence, the uniqueness theorem can be formulated separately for each of the classes W^p and W^s.

A different pattern is observed when we go over to problems for the exterior of bounded domains (the exterior of piecewise continuously differentiable closed curves). Here, because of the definite conditions on the boundary of the domain, both kinds of waves turn out ot be related in the general case, and solutions cannot exist in the class W^p or W^s taken separately.

Formulation of the uniqueness theorem for the exterior of a bounded domain G with boundary B (exterior problem of vibrations theory) will differ just by the fact that "in the exterior of the domain G" should be written everywhere in the conditions of the theorem in place of "in the whole x_1x_2 plane", and the following condition should be added:

e) The displacements (or stresses) equal zero on the boundary B of the domain G.

In conclusion, let us note that as in the case of the Sommerfeld conditions, the relationships (4.9) are the main components of conditions (4.1) and (4.2). The relation $u_i^l = O(r^{-1/2})$ indicates only that the solution decreases as $r^{-1/2}$ at infinity. There is an infinity of such solutions. The conditions mentioned correspond at infinity to diverging waves when there are no perturbation sources at infinity.

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